Binary Relations

A relation from a set A to a set B is a subset of $A \times B = \{(a,b) : a \in A, b \in B\}$.

Examples: The graph of a function $y = f(x)$ is a relation $F = \{(x, f(x)) : x \in \text{domain of } f\}$
$\subseteq \mathbb{R}^2$

This is a relation from $\mathbb{R}$ to $\mathbb{R}$.

A relation from A to A is just called a relation on A.

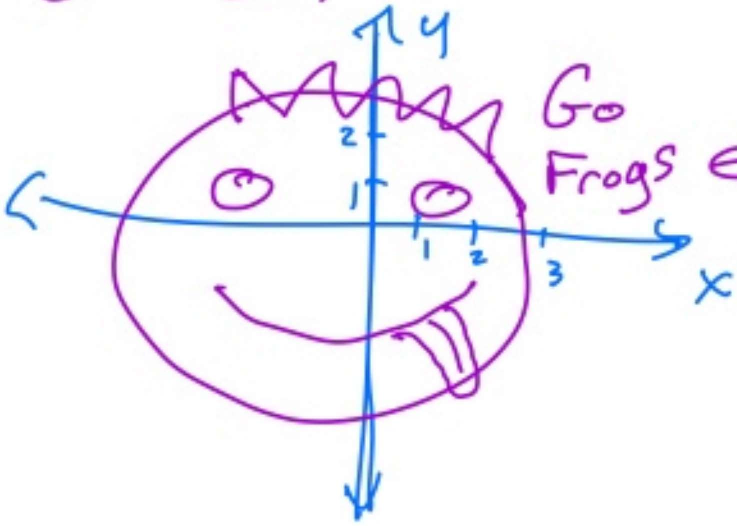
ie This is a relation on $\mathbb{R}$.

Other relations on $\mathbb{R}$:

to say (x,y) is part of the relation

- $(x,y) \in C$
- $x\,C\,y$

- $C = \{(x,y) : x^2+y^2=1\}$ Not a function, but it's a relation.

- 

a relation on $\mathbb{R}$.

- Any subset of $\mathbb{R}^2$ is a relation.

Other Relations

- P = set of all people
- One relation R on P: we say aRb for $a,b \in P$ if a & b have seen the same movie.

  eg Sumalee R Michelle.
- Another relation S on P: We say aSb if a's and b's first names end with the same letter.

  Thus $\text{Alex}_2 \, S \, \text{Alex}_3$ $\quad$ $\text{Alex}_3 \not S \, \text{Andrew}$

  $\text{Alex}_2 \not S \, \text{Nolan}$ $\quad$ Brae S Robbie

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Properties of Relations.

- We say a relation R on a set A is
  - <u>Reflexive</u> if $aRa \; \forall a \in A$.

    $(a,a) \in R \; \forall \; a \in A$.
  - <u>Symmetric</u> if whenever aRb for $a,b \in A$ then also bRa.

    If $(a,b) \in R$, then also $(b,a) \in R$.

★★★ <u>Transitive</u> if whenever

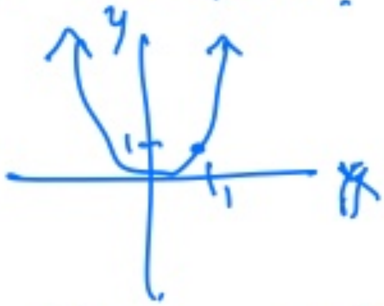
If $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

aRb and bRc for some $a,b,c \in A$, then also aRc.

An <u>equivalence relation</u> is a relation on a set that is reflexive, symmetric and transitive.

<u>Examples:</u> Let $P = \{(x, x^2) : x \in \mathbb{R}\}$.

P is a relation on $\mathbb{R}$.



Is P reflexive? No

i.e. is it true that $(x,x) \in P$ $\forall x \in \mathbb{R}$?

for example, $(2,2) \notin P$, and $2 \in \mathbb{R}$.

[A graph can be reflexive only iff it contains the whole line $y=x$.].

Is P symmetric?

i.e. is it true that if $(x,y) \in P$ then also $(y,x) \in P$?

No. Proof: $(2,4) \in P$ but $(4,2) \notin P$.

[A graph can be a symmetric relation iff

When you reflect across $y=x$, it is the same (ie switching x,y- axes does not change the graph.).]

Is P transitive?

No. Proof: $(2,4)\in P$, $(4,16)\in P$, but $(2,16)\notin P$. Thus P is not transitive.

So P is not an equivalence relation.

We could make P an equivalence relation by change P to

$$\tilde{P} = \{(x,y) : y=x \text{ or } y=x^2 \text{ or } x=y^2 \text{ or } y=x^4 \text{ or } x=y^4 \text{ or } y=x^8 \text{ or } x=y^8 \text{ or} \ldots\}$$

Then it would be an equivalence relation. (I think).

Relations on A are just subsets, so you can do set operations:

If R_1, R_2 are relations on A, then $R_1 \cap R_2$, $R_1 \cup R_2$, $R_1 \setminus R_2$, etc are also relations.

Compositions of relations:

If R is a relation from B to C and S is a relation from A to B then $R \circ S$ is a relation from A to C

defined by

$$R \circ S = \{(a, c) : aSb \text{ and } bSc \text{ for } a \in A, \text{ some } b \in B, c \in C\}.$$

Example. Let Q be the set of all people. Let C be the relation on Q be defined by

aCb if a and b are first cousins or the same person.

<u>Question</u>: is C reflexive, symmetric, transitive?

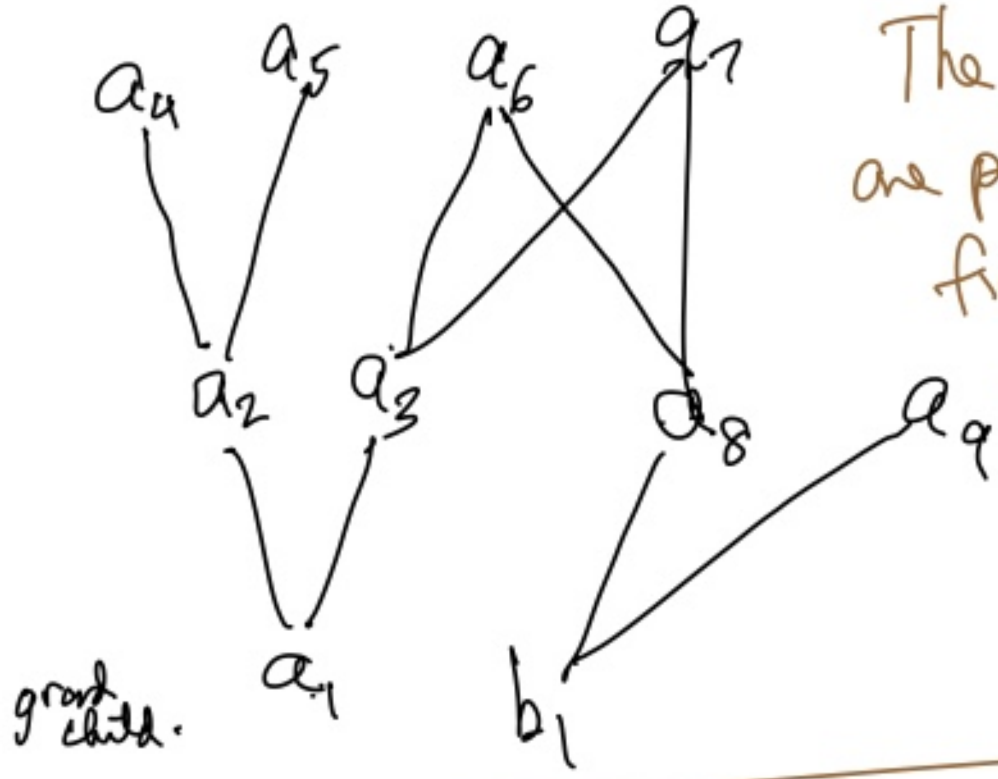
- <u>Reflexive</u>: Is it true that $\forall a \in Q$, aCa? Yes, because if $a \in Q$, then a is the same person as a.
- <u>Symmetric</u>: Is it true that $\forall a, b \in Q$, if aCb, then bCa?

  Yes, because if a is a first cousin of b, then b is a first cousin of a.

- <u>Transitive</u>: Is it true that $\forall a, b, c \in G$, if aCb and bCc, then aCc?

  <u>No</u>. For example, with a family tree like this.

  It just doesn't work. QED.



The first cousin on one parent side and the first cousin on the other parent side are usually not first cousins.